ON BINARY QUADRATIC FORMS AND THE HECKE GROUPS

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Abstract

We present a theory of reduction of binary quadratic forms with coefficients in $\mathbb{Z}[\lambda]$, where λ is the minimal translation in a Hecke group. We generalize from the modular group $\Gamma(1) = \mathrm{SL}(2,\mathbb{Z})$ to the Hecke groups and make extensive use of modified negative continued fractions. We also caracterize the "reduced" and "simple" hyperbolic fixed points of the Hecke groups.

The modular group and negative continued fractions play key roles in the development of a theory of reduction of binary quadratic forms with integer coefficients as given in [8]. In this paper we generalize from the modular group to the Hecke groups by replacing $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ with $\left(\begin{smallmatrix}1&\lambda\\0&1\end{smallmatrix}\right)$ for certain values of λ between 1 and 2. We use Rosen's negative λ-continued fractions [5], which are associated with the Hecke groups. We also make use of original work done in this direction by Schmidt and Sheingorn [6]. The result is a theory of reduction of indefinite binary quadratic forms with coefficients in any one of the rings $\mathbb{Z}[\lambda]$.

Part of the motivation for this investigation is our wish to uncover properties of $\mathbb{Z}[\lambda]$-binary quadratic forms which will be useful for characterizing rational period functions of automorphic integrals on the Hecke groups [1].

In Sections 1 and 2 we define the Hecke groups and λ-continued fractions. In Section 3 we introduce the idea of a hyperbolic $\mathbb{Z}[\lambda]$-binary quadratic form and describe a unique association between primitive indefinite forms and certain real numbers. In Section 4 we characterize the "reduced" hyperbolic points of the Hecke groups. Section 5 contains the main reduction theorem. In Section 6 we characterize "simple" forms and show that they may be put into cycles corresponding to equivalence classes.

1 HECKE GROUPS

In this section we give basic properties of the Hecke groups. We use the properties of some elements to define a family of functions which we will use later to produce cycles of "simple" binary quadratic forms.

1.1 DEFINITIONS

Let $S = S_\lambda = \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$, $T = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, where λ is a positive real number. Put $G(\lambda) = \langle S, T\rangle/\{\pm I\} \subseteq \mathrm{SL}(2,\mathbb{R})$. Erich Hecke [2] showed that the only values of λ for which $G(\lambda)$ is discrete are

$$\lambda = \lambda_p = 2\cos(\pi/p),$$

for $p = 3, 4, 5, \ldots$, and $\lambda \geq 2$. We will focus on the discrete groups with $\lambda < 2$, *i.e.*, those with $\lambda = \lambda_p$, $p \geq 3$. These groups have come to be known as the *Hecke groups*, and we will denote them by $G_p = G(\lambda_p)$ for $p \geq 3$. The first several of these Hecke groups are $G_3 = G(1) = \Gamma(1)$ (the modular group), $G_4 = G(\sqrt{2})$, $G_5 = G\left(\frac{1+\sqrt{5}}{2}\right)$, and $G_6 = G(\sqrt{3})$.

Fix $p \geq 3$ and let $U = U_{\lambda_p} = S_{\lambda_p}T \in G_p$. The generators of G_p satisfy the two relations

$$T^2 = U^p = I.$$

The groups with $\lambda \geq 2$ have only one relation $T^2 = I$.

The entries of elements of G_p are in $\mathbb{Z}[\lambda_p]$, which for each $p \geq 3$ is the ring of algebraic integers for $\mathbb{Q}(\lambda_p)$. For $p > 3$, $\mathbb{Q}(\lambda_p)$ has nontrivial units and may have a nontrivial class group. For example, $h_{\mathbb{Q}(\lambda_{68})} = 2$. This is known to be the only class number greater than 1 for $p \leq 73$ [7].

For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_p$, we have $ad-bc = 1$, so G_p is a subgroup of $\mathrm{SL}(2,\mathbb{Z}[\lambda_p])$. It is well-known that $G_3 = \mathrm{SL}(2,\mathbb{Z}[\lambda_3])$ (*i.e.*,$\Gamma(1) = \mathrm{SL}(2,\mathbb{Z})$), however for the other Hecke groups $G_p \subsetneqq \mathrm{SL}(2,\mathbb{Z}[\lambda_p])$.

An element $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_p$ is *hyperbolic* if $|a+d| > 2$, *parabolic* if $|a+d| = 2$, and *elliptic* if $|a+d| < 2$. Elements of the Hecke group act on $\mathbb{C}$ as linear fractional transformations. A complex number z is a *fixed point* of $M \in G_p$ if $Mz = z$, so $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ fixes

$$\begin{aligned} z &= \frac{a-d \pm \sqrt{(d-a)^2 + 4bc}}{2c} \\ &= \frac{a-d \pm \sqrt{(a+d)^2 - 4}}{2c}. \end{aligned} \tag{1}$$

From this it is clear that hyperbolic elements of G_p each have two distinct real fixed points. Parabolic elements each have one real fixed point and elliptic elements each have non real fixed points which are complex conjugates of each other.

Since G_p is discrete, the *stabilizer* of any complex number z in G_p, $\{M \in G_p \mid Mz = z\}$ is a cyclic subgroup of G_p [3, page 15]. Thus the fixed point sets of any two elements of G_p are identical or disjoint, and all elements of a stabilizer have identical fixed points. Accordingly, we may designate fixed points as *hyperbolic*, *parabolic*, or *elliptic* according to whether the matrices fixing them are hyperbolic, parabolic, or elliptic, respectively. We also define the *Hecke conjugate* of any hyperbolic fixed point to be the other fixed point of the elements in its stabilizer.

1.2 THE MAP Φ_p

We need the following Lemma in order to define a function we will use in Section 6. The main idea of the proof will also be useful later. We thank the referee for simplifying the proof.

Lemma 1. *Fix $p \geq 3$ and let $U = U_{\lambda_p}$. Suppose that $\alpha \in \mathbb{R} \cup \{\infty\}$ and $1 \leq i \leq p-1$. Then $\alpha \in \big[U^{p-i+1}(0), U^{p-i}(0)\big)$ if and only if $TU^i(\alpha) \in [0,\infty)$, and $\alpha = U^{p-i+1}(0)$ if and only if $TU^i(\alpha) = 0$.*

Proof. A calculation shows that

$$0 = U^p(0) < U^{p-1}(0) < \cdots < U^2(0) < U(0) = \infty,$$

so the intervals make sense, and

$$[-\infty,\infty) = [-\infty, U^p(0)) \cup [U^p(0), U^{p-1}(0)) \cup \cdots \cup [U^2(0), U(0)) \qquad (2)$$

is a disjoint union of p half-open intervals. U maps each interval to the previous interval, the first to the last, and left endpoints to left endpoints. Thus $\alpha \in \big[U^{p-i+1}(0), U^{p-i}(0)\big) = \big[U^{-i+1}(0), U^{-i}(0)\big)$ if and only if $U^i(\alpha) \in [-\infty, 0)$, which is true if and only if $TU^i(\alpha) \in [0,\infty)$, with left endpoints corresponding to left endpoints. $\square$

By the proof of Lemma 1, there is a $(p-1)$ to one function Φ_p mapping $[0,\infty)$ to $[0,\infty)$ defined by

$$\Phi_p(x) = \begin{cases} TUx, & U^p(0) \leq x < U^{p-1}(0) \\ TU^2x, & U^{p-1}(0) \leq x < U^{p-2}(0) \\ \vdots & \\ TU^{p-1}x, & U^2(0) \leq x, \end{cases}$$

where $U = U_{\lambda_p}$. This function is given more explicitly by

$$\Phi_p(x) = \begin{cases} \frac{x}{1-\lambda x}, & 0 \leq x < 1/\lambda \\ \frac{1-\lambda x}{(\lambda^2-1)x-\lambda}, & 1/\lambda \leq x < \lambda/(\lambda^2-1) \\ \vdots & \\ x-\lambda, & \lambda \leq x, \end{cases}$$

where $\lambda = \lambda_p$. Given $\alpha \in [0, \infty)$, the exponent i in $\Phi_p(\alpha) = TU^i\alpha$ is the unique exponent between 1 and $p-1$ for which $TU^i\alpha \in [0, \infty)$.

2 CONTINUED FRACTIONS

In this section we define λ_p-continued fractions. We also derive some properties of these continued fractions for later use.

Rosen [5] introduced a class of continued fractions closely associated with the Hecke groups. He expanded any real number into a continued fraction using a *nearest* integral multiple of λ algorithm. We will use a modification of Rosen's continued fractions, in which we expand real numbers into continued fractions using a *next* integral multiple of λ algorithm. The result will be *negative* (or *backwards*) continued fractions, in which every numerator is -1. These continued fractions, also used in [6], generalize the simple negative continued fractions in [8] for which $\lambda = 1$ and $p = 3$. They will help us make connections between poles of RPFs, elements of G_p, and binary quadratic forms on $\mathbb{Z}[\lambda_p]$.

Fix $p \geq 3$, put $\lambda = \lambda_p$ and let $r_j \in \mathbb{Z}$ for $j \geq 0$. Define a *finite λ_p-continued fraction* (λ_p-CF or λ-CF) by

$$\begin{aligned} [r_0; r_1, \ldots, r_n] &= r_0\lambda - \cfrac{1}{r_1\lambda - \cfrac{1}{\ddots - \cfrac{1}{r_n\lambda}}} \\ &= (S^{r_0}TS^{r_1}T\ldots S^{r_n}T)(\infty). \end{aligned}$$

We define an *infinite* λ_p-CF by

$$[r_0; r_1, \ldots] = \lim_{n\to\infty} [r_0; r_1, \ldots, r_n],$$

if the limit exists.

We expand any finite real number α into a λ-CF according to the next integral multiple of λ algorithm. Let $\alpha_0 = \alpha$ and for $j \geq 0$ define

$$r_j = \left[\frac{\alpha_j}{\lambda}\right] + 1, \tag{3}$$

and the $j + 1^{\text{st}}$ *complete quotient*

$$\alpha_{j+1} = \frac{1}{r_j\lambda - \alpha_j}. \tag{4}$$

Here $[\,\cdot\,]$ is the greatest integer function. Then $\alpha_j = r_j\lambda - \frac{1}{\alpha_{j+1}}$ for $j \geq 0$ and $[r_0; r_1, \ldots]$ is the unique λ-CF for α, while $[r_j; r_{j+1}, \ldots]$ is the unique λ-CF for α_j. We note that (3) and (4) imply that for $j \geq 1$ we have $r_j \geq 1$ and $\alpha_j \geq \frac{1}{\lambda}$.

We define an *admissible* λ-CF to be one that arises from a finite real number by the next integral multiple of λ algorithm. Then we have

Lemma 2. *Fix $p \geq 3$ and put $\lambda = \lambda_p$. Then every admissible λ-CF converges.*

Proof. Put $S = S_\lambda$ and let $[r_0; r_1, \ldots]$ be the admissible λ-CF for $\alpha \in \mathbb{R}$. For any $n \geq 0$ we define $C_n = [r_0; r_1, \ldots, r_n]$, the *nth convergent* of the λ-CF. We will show that $\{C_n\}_{n=0}^{\infty}$ is decreasing and bounded below by α.

We define $C_{m,n} = [r_m; r_{m+1}, \ldots, r_n]$ for $0 \leq m \leq n$ and note that $C_{n,n} = r_n\lambda > \alpha_n$. For all $n > 0$, $\alpha_n > 0$ and $C_{n-1,n} = S^{r_{n-1}}T\left(C_{n,n}\right) > S^{r_{n-1}}T\left(\alpha_n\right) = \alpha_{n-1}$, since $S^jT(x) = j\lambda - \frac{1}{x}$ increases monotonically on $(0, +\infty)$. Continuing, we have that $C_{m,n} > \alpha_m$ for $0 \leq m \leq n$. In particular, $C_n = C_{0,n} > \alpha_0 = \alpha$ for all $n \geq 0$.

In order to show that $\{C_n\}_{n=0}^{\infty}$ is decreasing, we fix $n \geq 0$ and note that $C_{n,n} = r_n\lambda > r_n\lambda - \frac{1}{r_{n+1}\lambda} = C_{n,n+1}$. Then for $n > 0$, $C_{n-1,n} = S^{r_{n-1}}T\left(C_{n,n}\right) > S^{r_{n-1}}T\left(C_{n,n+1}\right) = C_{n-1,n+1}$. Continuing, we have that $C_{m,n} > C_{m,n+1}$ for all m, $0 \leq m \leq n$. In particular, $C_n > C_{n+1}$ for all $n \geq 0$. $\square$

A *periodic* λ_p-CF is an infinite λ-CF that repeats, *i.e.*, one for which there exist $n \geq 0$ and $m \geq 1$ such that $r_{j+m} = r_j$ for all $j \geq n$. We will take n and m to be the smallest integers for which this happens, and write a periodic λ-CF as

$$[r_0; r_1, \ldots, r_{n-1}, \overline{r_n, \ldots, r_{n+m-1}}].$$

A *purely periodic* λ-CF has $n = 0$.

Periodic λ-CFs identify the fixed points of G_p. Indeed, we have the following result of Schmidt and Sheingorn [6].

Lemma 3. *A real number is a fixed point of G_p, $p \geq 3$, if and only if it has a periodic λ_p-CF expansion. Moreover, such a number is parabolic if and only if its λ_p-CF has the period $[\overline{2, \underbrace{1, \ldots, 1}_{p-3}}]$, and is hyperbolic if and only if its λ_p-CF has a period other than $[\overline{2, \underbrace{1, \ldots, 1}_{p-3}}]$.*

Proof. This is contained in Lemmas 1, 2, and 3 in [6]. $\square$

There are restrictions on admissible λ-CFs. We have

Lemma 4. *Put $p \geq 3$. An admissible λ_p-CF*

(i) has at most $p-3$ consecutive ones in any position but the beginning, and

(ii) has at most $p-2$ consecutive ones at the beginning.

Remark. In the classical case ($p = 3$ and $\lambda = 1$) Lemma 4 reduces to the fact that admissible simple negative continued fractions have no ones in any position. This agrees with the theory of simple negative continued fractions as developed in [8].

Proof. Set $\lambda = \lambda_p$, $S = S_\lambda$ and $U = U_\lambda$, and let $\alpha \in \mathbb{R}$. For any $m \geq 0$ and $j \geq 1$ we have that $\alpha_m = S^{r_m} T S^{r_{m+1}} T \cdots S^{r_{m+j-1}} T \alpha_{m+j}$. If the λ-CF has j consecutive ones starting with r_m, then $\alpha_m = U^j \alpha_{m+j}$. By the next multiple of λ algorithm we have that for any $n \geq 1$, $\alpha_n \in [1/\lambda, \infty)$, which is the union of the last $p-2$ intervals in (2). Thus for $m \geq 1$ we must have $j \leq p-3$. Otherwise $\alpha_{m+k} \notin [1/\lambda, \infty)$ for some k, $1 \leq k \leq j$, since U maps each interval to the previous interval in (2). $\square$

The restrictions in Lemma 4 are the best possible, since for any $\lambda = \lambda_p$, $p \geq 3$, $\frac{3}{2}\lambda + \frac{1}{2}\sqrt{\lambda^2+4} = [\overline{3; \underbrace{1, \ldots, 1}_{p-3}}]$ and $(ST)^{-2}\,[\overline{3; \underbrace{1, \ldots, 1}_{p-3}}] = [1, \overline{\underbrace{1, \ldots, 1}_{p-3}, 3}]$.

3 BINARY QUADRATIC FORMS

We will exploit the connections between binary quadratic forms over $\mathbb{Z}[\lambda_p]$, elements of the Hecke group $G_p = G(\lambda_p)$, and λ_p-CFs.

We consider binary quadratic forms with coefficients in $\mathbb{Z}[\lambda_p]$,

$$Q(x, y) = Ax^2 + Bxy + Cy^2.$$

We denote such a form by $Q = [A, B, C]$ and refer to it as a λ_p-BQF. We restrict our attention to indefinite forms, which have positive discriminant $D = B^2 - 4AC$.

3.1 FORMS AND NUMBERS

We associate the λ_p-BQF $Q = [A, B, C]$ with the number $\alpha_Q = \frac{-B+\sqrt{D}}{2A} \in \mathbb{Q}(\lambda_p, \sqrt{D})$, one of the zeros of $Q(x) = Q(x, 1)$. Under this association, the form $-Q = [-A, -B, -C]$ maps to the other zero of $Q(x)$, $\alpha_{-Q} = \frac{B+\sqrt{D}}{-2A} = \frac{-B-\sqrt{D}}{2A}$. This association is ambiguous in the other direction. However, if we restrict out attention to hyperbolic fixed points, we may describe a unique association of number to forms. We thank the referee for providing a shorter proof of the following Lemma.

Lemma 5. *Every hyperbolic fixed point of G_p, $p \geq 3$, may be associated with a unique indefinite $\mathbb{Z}[\lambda_p]$-binary quadratic form.*

Proof. Suppose that $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $M^{-1} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$ generate the stabilizer of hyperbolic fixed point α in G_p. Now $M\alpha = \alpha$ gives rise to the equation

$$c\alpha^2 + (d - a)\alpha - b = 0,$$

and $M^{-1}\alpha = \alpha$ gives rise to the equation

$$-c\alpha^2 - (d - a)\alpha - (-b) = 0.$$

We put

$$M_\alpha = \begin{cases} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) & \text{if } \alpha = \frac{a-d+\sqrt{D}}{2c}, \\ \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right) & \text{if } \alpha = \frac{a-d-\sqrt{D}}{2c}, \end{cases}$$

and

$$Q_\alpha = \begin{cases} [c, d - a, -b] & \text{if } \alpha = \frac{a-d+\sqrt{D}}{2c}, \\ -[c, d - a, -b] & \text{if } \alpha = \frac{a-d-\sqrt{D}}{2c}, \end{cases}$$

where $D = (d - a)^2 + 4bc = (a + d)^2 - 4$. □

If a λ-BQF Q arises from a hyperbolic fixed point as in the proof of Lemma 5, we say that Q is *hyperbolic*.

It is easy to calculate the generators of the stabilizer of any fixed point α in G_p. By Lemma 3, α has a periodic λ_p-CF expansion $\alpha = [r_0; r_1, \ldots, r_{n-1}, \overline{r_n, \ldots, r_{n+m-1}}]$. We put $V = S^{r_0}TS^{r_1}T \cdots S^{r_{n-1}}T$ and $W = S^{r_n}TS^{r_{n+1}}T \cdots S^{r_{n+m-1}}T$. Then $M = VWV^{-1}$ and $M^{-1} = VW^{-1}V^{-1}$ generate the stabilizer of α in G_p since every λ_p-CF period is minimal.

Suppose that $\alpha = \frac{-B+\sqrt{D}}{2A}$ is a hyperbolic fixed point of G_p associated with the hyperbolic λ_p-BQF $Q_\alpha = [A, B, C]$. We denote the Hecke conjugate of α by α' and observe that by the proof of Lemma 5, $\alpha' = \frac{-B-\sqrt{D}}{2A}$ is associated with $Q_{\alpha'} = -Q_\alpha = [-A, -B, -C]$. A straightforward calculation shows that if $V \in G_p$, then $(V\alpha)' = V\alpha'$.

3.2 EQUIVALENCE CLASSES

Elements of a Hecke group act on λ-BQFs by $(Q \circ M)(x, y) = Q(ax+by, cx+dy)$ for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_p$. More explicitly, if $Q = [A, B, C]$ we have that

$$[A, B, C] \circ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = [A', B', C'],$$

where

$$A' = Aa^2 + Bac + Cc^2 = Q(a, c),$$

$$B' = 2Aab + B(ad + bc) + 2Ccd,$$

and

$$C' = Ab^2 + Bbd + Cd^2 = Q(b,d).$$

A straightforward calculation shows that $B'^2 - 4A'C' = B^2 - 4AC$, so the action of a Hecke group preserves the discriminant.

We say that Q and Q' are G_p-*equivalent*, and write $Q \sim Q'$, if there exists a $V \in G_p$ such that $Q' = Q \circ V$. It is easy to check that G_p-equivalence is an equivalence relation, so G_p partitions the λ-BQFs into equivalence classes of forms.

Lemma 6. *Fix $p \geq 3$ and let $\lambda = \lambda_p$. Suppose that Q_α and Q_β are hyperbolic λ-BQFs associated with hyperbolic numbers α and β, respectively, and let $V \in G_p$. Then $Q_\beta = Q_\alpha \circ V$ if and only if $\beta = V^{-1}\alpha$.*

Proof. We write $Q_\alpha = [A, B, C]$, so $\alpha = \frac{-B+\sqrt{D}}{2A}$, where $D = B^2 - 4AC$, and we write $V = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Suppose that $Q_\beta = Q_\alpha \circ V$. Then

$$\begin{aligned} Q_\beta &= [A, B, C] \circ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \\ &= [Aa^2 + Bac + Cc^2, 2Aab + B(ad + bc) + 2Ccd, Ab^2 + Bbd + Cc^2], \end{aligned}$$

so

$$\beta = \frac{-(2Aab + B(ad + bc) + 2Ccd) + \sqrt{D}}{2(Aa^2 + Bac + Cc^2)}.$$

On the other hand, $V^{-1} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$, so

$$\begin{aligned} V^{-1}\alpha &= \frac{d\left(\frac{-B+\sqrt{D}}{2A}\right) - b}{-c\left(\frac{-B+\sqrt{D}}{2A}\right) + a} \\ &= \frac{-2Ab - Bd + c\sqrt{D}}{2Aa - Bc - c\sqrt{D}} \\ &= \frac{-(2Aab + B(ad + bc) + 2Ccd) + \sqrt{D}}{2(Aa^2 + Bac + Cc^2)} \\ &= \beta. \end{aligned}$$

Now suppose that $\beta = V^{-1}\alpha$. Let M be one of the two generators of the stabilizer of α in G_p. Then $W = V^{-1}MV$ is one of the two generators of the stabilizer of β in G_p. Moreover, M and W have the same trace, since the trace of a matrix is preserved by conjugation. Now $M\alpha = \alpha$ gives rise to Q_α and $W\beta = \beta$ gives rise to Q_β as in the proof of Lemma 5, so Q_α and Q_β have the same discriminant. By the calculation above, $Q_\alpha \circ V$ has the associated hyperbolic number

$$\frac{-(2Aab + B(ad + bc) + 2Ccd) + \sqrt{D}}{2(Aa^2 + Bac + Cc^2)} = V^{-1}\alpha = \beta.$$

A similar calculation shows that $-(Q_\alpha \circ V)$ has the associated hyperbolic number

$$\frac{-(2Aab + B(ad+bc) + 2Ccd) - \sqrt{D}}{2(Aa^2 + Bac + Cc^2)} = V^{-1}\alpha' = \beta'.$$

Thus $(Q_\alpha \circ V)(x, 1)$ and $Q_\beta(x, 1)$ have the same discriminant and the same zeros, so they are equal. Hence $Q_\alpha \circ V = Q_\beta$. □

Lemma 6 implies that G_p-equivalence of hyperbolic λ-BQFs induces a corresponding G_p-equivalence of associated numbers. This equivalence yields the following.

Lemma 7. *Two hyperbolic fixed points of G_p, $p \geq 3$, are equivalent if and only if the periods of their λ_p-CF expansions are cyclic permutations of each other.*

Corollary. *Let $p \geq 3$ and put $\lambda = \lambda_p$. Every G_p-equivalence class of λ-BQFs contains either all hyperbolic forms or no hyperbolic forms.*

Proof. G_p-equivalence classes of hyperbolic λ_p-BQFs are in $1-1$ correspondence with G_p-equivalence classes of hyperbolic fixed points of G_p. The Corollary follows from Lemma 3 and Lemma 7. □

We may now label equivalence classes themselves as hyperbolic or non-hyperbolic.

4 REDUCED NUMBERS

In this section we generalize the reduction theory of indefinite binary quadratic forms to λ-BQFs. We characterize "reduced" numbers and we see that every hyperbolic G_p-equivalence class of forms contains a cycle of "reduced" forms. From this it follows that there are a finite number of hyperbolic equivalence classes for each discriminant.

Fix $p \geq 3$. We say that a real number α is a G_p*-reduced* number if the λ_p-CF expansion of α is purely periodic, with period other than $[\overline{2, \underbrace{1, \ldots, 1}_{p-3}}]$. If α is G_p-reduced, we also say that the associated λ_p-BQF Q_α is G_p-reduced.

The following Lemma is a modification of a familiar result from classical continued fractions. It is also stated in [6, page 389].

Lemma 8. *Let $p \geq 3$ and put $\lambda = \lambda_p$. If $\alpha = [\overline{r_0; r_1, \ldots, r_n}]$ is a λ-CF, then* $\frac{1}{\alpha'} = [\overline{r_n; r_{n-1}, \ldots, r_0}]$.

Proof. Put $S = S_\lambda$. We have $\alpha_j = [\overline{r_j; r_{j+1}, \ldots}]$ for $j \geq 0$. Then $\alpha_{j+1} = TS^{-r_j}\alpha_j$, and $\alpha_0 = \alpha_{n+1} = TS^{-r_n}\alpha_n$. Taking Hecke conjugates, we have

$\alpha'_{j+1} = TS^{-r_j}\alpha'_j = \frac{1}{r_j\lambda - \alpha'_j}$, $j \geq 0$, and $\alpha'_0 = \frac{1}{r_n\lambda - \alpha'_n}$, so $\frac{1}{\alpha'_0} = r_n\lambda - \alpha'_n$. We combine these to get

$$\begin{aligned}
\frac{1}{\alpha'_0} &= r_n\lambda - \cfrac{1}{r_{n-1}\lambda - \cfrac{1}{r_{n-2}\lambda - \ddots}} \\
&= [\overline{r_n; r_{n-1}, \ldots, r_0}].
\end{aligned}$$

□

We need the following result in the proof of the next Theorem.

Lemma 9. *Fix $p \geq 3$ and let $U = U_{\lambda_p}$. Then $\frac{1}{U^k(0)} = U^{p-k+1}(0)$ for any integer k.*

Proof. Let $c_k = \frac{sin(k\pi/p)}{sin(\pi/p)}$ for $k \geq 0$. Meier and Rosenberger [4] show that as linear fractional transformations

$$U^k = \left(\begin{smallmatrix} c_{k+1} & -c_k \\ c_k & -c_{k-1} \end{smallmatrix}\right), \tag{5}$$

for $k \in \mathbb{Z}^+$. In fact, it is easy to show that (5) holds for all integers k. Then

$$\begin{aligned}
U^{p-k+1}(0) &= \frac{c_{p-k+1}}{c_{p-k}} \\
&= \frac{sin\left((p-k+1)\pi/p\right)}{sin\left((p-k)\pi/p\right)} \\
&= \frac{sin\left((k-1)\pi/p\right)}{sin\left(k\pi/p\right)} \\
&= \frac{c_{k-1}}{c_k} \\
&= \frac{1}{U^k(0)}.
\end{aligned}$$

□

We next characterize G_p-reduced numbers. We thank the referee for making suggestions which helped to correctly formulate and prove the following Theorem.

Theorem 1. *Put $p \geq 3$, $\lambda = \lambda_p$, and $U = U_\lambda$. Suppose that α is a hyperbolic fixed point of G_p. Then α is G_p-reduced with k leading ones in its λ-CF if and only if k is the smallest nonnegative integer such that*

$$0 < \alpha' < U^{k+2}(0) < \alpha < U^{k+1}(0). \tag{6}$$

Remark. In the classical case ($p = 3$ and $\lambda = 1$) we must have that $k = 0$. Then a hyperbolic point α is ($\Gamma(1)-$) reduced if and only if $0 < \alpha' < 1 < \alpha < \infty$. This agrees with the definition of a reduced number in [8, page 128].

Proof. First suppose that α is G_p-reduced and that the λ-CF for α has k leading ones. Let $m \in \mathbb{Z}^+$ denote the period length of the λ-CF for α. Then $k \leq \min\{m-1, p-3\}$. We may write $\alpha = [\overline{r_0; r_1, \ldots, r_{m-1}}] = [\overline{\underbrace{1; 1, \ldots, 1}_{k}, r_k, r_{k+1}, \ldots, r_{m-1}}]$, and $\alpha_j = [\overline{r_j; r_{j+1}, \ldots, r_{m-1}, r_0, \ldots, r_{j-1}}]$ for $0 \leq j \leq m-1$. Each α_j is hyperbolic and thus not a multiple of λ. Since $r_k \geq 2$ we have $\alpha_k > \lambda = U^2(0)$, *i.e.*, α_k is in the pth interval of (2). Now $r_j = 1$ for $0 \leq j < k$ implies that $\alpha = U^k \alpha_k$, so α is in the $(p-k)$th interval of (2), *i.e.*, $U^{k+2}(0) < \alpha < U^{k+1}(0)$. By Lemma 8 we have $\frac{1}{\alpha'} = [\overline{r_{m-1}; \ldots, r_{k+1}, r_k, \underbrace{1, \ldots, 1}_{k}}] > 0$, so $\alpha' > 0$. Also, $\alpha_k = [\overline{r_k; r_{k+1}, \ldots, r_{m-1}, \underbrace{1, \ldots, 1}_{k}}]$, so $\frac{1}{\alpha'_k} = [\overline{\underbrace{1; 1, \ldots, 1}_{k}, r_{m-1}, \ldots, r_{k+1}, r_k}] > \frac{1}{\lambda} = U^{p-1}(0)$. Then $\frac{1}{\alpha'} = U^{-k}\left(\frac{1}{\alpha'_k}\right) > U^{p-k-1}(0) > 0$, since U^{-k} maps every interval in (2) k intervals to the right, and $2 \leq p-k-1 \leq p-1$. Thus $\alpha' < \frac{1}{U^{p-k-1}(0)} = U^{k+2}(0)$, by Lemma 9, and we have verified (6).

Next, suppose that k is the smallest nonnegative integer such that (6) holds. Since $U^p = I$, we have that $k \leq p-1$. Furthermore $k \leq p-2$, since $k = p-1$ implies that $\infty = U^{p+1}(0) < U^p(0) = 0$.

By (6), α is in the $p-k^{\text{th}}$ interval from the left and α' is in one of the second through the $p-k-1^{\text{st}}$ intervals of (2). Let $\alpha = [r_0; r_1, \ldots, r_{n-1}, \overline{r_n, \ldots, r_{n+m-1}}]$ denote the $\lambda-$CF for α. A complete quotient α_j is in an interval other than the p^{th} interval when $0 < \alpha_j < \lambda$ and $r_j = 1$. In this case $\alpha_{j+1} = U^{-1}\alpha_j$ and $\alpha'_{j+1} = U^{-1}\alpha'_j$, so α_{j+1} is in the interval to the right of α_j and α'_{j+1} is in the interval to the right of α'_j.

Since α is in the $p-k^{\text{th}}$ interval from the left, the discussion above implies that each α_j, $0 \leq j \leq k-1$, is in an interval to the left of the p^{th} interval, while α_k is in the p^{th} interval. Furthermore $r_j = 1$ for $0 \leq j \leq k-1$ and $r_k \geq 2$, *i.e.*, the λ-CF for α has k leading ones.

In order to show that α is G_p-reduced, we first claim that $0 < \alpha'_j < \lambda$ for all $j \geq 0$. Since α' is in one of the second through the $p-k-1^{\text{st}}$ intervals of (2), and the λ-CF for α has k leading ones, the discussion above implies that each α'_j, $0 \leq j \leq k$ is in one of the second through the $p-1^{\text{st}}$ intervals, *i.e.*, $0 < \alpha'_j < \lambda$ for $0 \leq j \leq k$. A calculation shows that if $0 < \alpha'_t < \lambda$ and $r_t \geq 2$ for any t, then $0 < \alpha'_{t+1} < \frac{1}{\lambda}$ and α'_{t+1} is in the second interval of (2). If r_t is followed by $r_{t+1} \geq 2$, then $0 < \alpha'_{t+2} < \frac{1}{\lambda}$ by the same calculation. On the other hand, if r_t is followed by ℓ ones, α'_j is in one of the 2^{nd} through the $\ell+2^{\text{nd}}$ intervals for $t+1 \leq j \leq t+\ell+1$. Since the λ-CF for α'_j is admissible, $\ell \leq p-3$, so each such α'_j is in one of the 2^{nd} through the $p-1^{\text{st}}$ intervals. Thus $0 < \alpha'_j < \lambda$ for the ℓ complete quotients following α_t. The claim follows by induction on j.

We next note that for every $j \geq 1$, the complete quotient α_j has a unique predecessor α_{j-1} with $0 < \alpha'_j < \lambda$. Indeed, $0 < \alpha'_j < \lambda$ implies that $T\alpha'_j < 0$, so $0 < S^t T \alpha'_j < \lambda$ for a unique t, $t \geq 1$. But since $0 < \alpha'_{j-1} < \lambda$ and $\alpha'_{j-1} = S^{r_{j-1}} T \alpha'_j$, we must have that $r_{j-1} = t$ is uniquely determined. Then $\alpha_n = \alpha_{n+m}$ implies that $\alpha_j = \alpha_{j+m}$ for every $j < n$ and $\alpha = [\overline{r_0; r_1, \ldots, r_{m-1}}]$. ∎

5 REDUCTION

We are ready to describe the reduction of hyperbolic elements of G_p.

Theorem 2. *Let $p \geq 3$ and put $\lambda = \lambda_p$. Every hyperbolic fixed point α of G_p may be transformed into a G_p-reduced number by finitely many applications of TS_λ^{-r}, where at each step $r = \left[\frac{\alpha}{\lambda}\right] + 1$. Furthermore, TS_λ^{-r} maps reduced numbers to reduced numbers, so the reduced numbers fall into disjoint cycles. Finally, every equivalence between G_p-reduced numbers is obtained by iteration of TS_λ^{-r}; thus each equivalence class of hyperbolic fixed points contains one cycle of G_p-reduced numbers.*

Proof. Suppose that $\alpha = \alpha_0$ is a hyperbolic fixed point of G_p. Then

$$\alpha_0 = [r_0; r_1, \ldots, r_{n-1}, \overline{r_n, \ldots, r_{n+m-1}}],$$

$r_0 = \left[\frac{\alpha_0}{\lambda}\right] + 1$. Put

$$\begin{aligned} \alpha_1 &= TS^{-r_0}\alpha_0 \\ &= [r_1; r_2, \ldots, r_{n-1}, \overline{r_n, \ldots, r_{n+m-1}}], \end{aligned}$$

with $S = S_\lambda$. We repeat this process, at each step using the mapping TS^{-r_j}, $r_j = \left[\frac{\alpha_j}{\lambda}\right] + 1$ to calculate α_{j+1}. After n steps, we have

$$\begin{aligned} \alpha_n &= TS^{-r_{n-1}}\alpha_{n-1} \\ &= [\overline{r_n; \ldots, r_{n+m-1}}], \end{aligned}$$

which is reduced.

Next, suppose that $\beta = \beta_0$ is a reduced number, so

$$\beta_0 = [\overline{r_0; r_1, \ldots, r_n}],$$

$r_0 = \left[\frac{\beta_0}{\lambda}\right] + 1$. Then

$$\begin{aligned} \beta_1 &= TS^{-r_0}\beta_0 \\ &= [\overline{r_1; r_2, \ldots, r_n, r_0}], \end{aligned}$$

and β_1 is also reduced. Repeating this process yields a cycle of n reduced numbers. Since each mapping is uniquely determined, cycles of reduced numbers must be disjoint.

The remaining statements in the Theorem follow from Lemma 7. $\square$

The proof of Theorem 2 makes it clear that there is a $1-1$ correspondence between cycles of reduced forms and admissible periods of hyperbolic numbers. We restate Theorem 2 for reduction of hyperbolic λ-BQFs.

Corollary. *Let $p \geq 3$ and put $\lambda = \lambda_p$. Every hyperbolic λ-BQF Q may be transformed into a reduced λ-BQF by finitely many applications of $S_\lambda^r T$, where at each step $r = \left[\frac{\alpha_Q}{\lambda}\right] + 1$. Furthermore, $S_\lambda^r T$ maps reduced forms to reduced forms, so the reduced forms fall into disjoint cycles. Finally, every equivalence between reduced λ-BQFs is obtained by iteration of $S_\lambda^r T$; thus each equivalence class of λ-BQFs with hyperbolic forms contains one cycle of reduced forms.*

Example 1. Put $\lambda = \lambda_5 = \frac{1+\sqrt{5}}{2}$, $S = S_\lambda$, and let $\alpha_0 = [2; 3, \overline{2, 1, 1, 4}]$. Then $M_{\alpha_0} = \left(\begin{smallmatrix} -9\lambda-6 & 51\lambda+32 \\ -3\lambda-2 & 18\lambda+9 \end{smallmatrix}\right)$ generates the stabilizer of α_0, and $Q_{\alpha_0} = [-3\lambda-2, 27\lambda+15, -51\lambda-32]$ is the λ_5-BQF corresponding to α_0. Now Q_{α_0} is in a hyperbolic equivalence class $\mathcal{A}$ of forms of discriminant $D = 135\lambda + 86 = \frac{307+135\sqrt{5}}{2}$. We reduce Q_{α_0} by

$$\begin{aligned} Q_{\alpha_1} &= Q_{\alpha_0} \circ S^2 T = [\lambda+2, -7\lambda-3, -3\lambda-2], \\ Q_{\alpha_2} &= Q_{\alpha_1} \circ S^3 T = [3\lambda+4, -11\lambda-3, \lambda+2], \end{aligned}$$

which is reduced. The cycle of reduced λ_5-BQFs in $\mathcal{A}$ is

$$\begin{aligned} Q_{\alpha_3} &= Q_{\alpha_2} \circ S^2 T = [13\lambda+8, -17\lambda-9, 3\lambda+4], \\ Q_{\alpha_4} &= Q_{\alpha_3} \circ ST = [11\lambda+8, -25\lambda-17, 13\lambda+8], \\ Q_{\alpha_5} &= Q_{\alpha_4} \circ ST = [\lambda+2, -13\lambda-5, 11\lambda+8], \text{and} \\ Q_{\alpha_2} &= Q_{\alpha_5} \circ S^4 T = [3\lambda+4, -11\lambda-3, \lambda+2]. \end{aligned}$$

The successive values of r in $S^r T$ are the successive entries in the λ_5-CF for α_0. The corresponding reduced numbers are

$$\begin{aligned} \alpha_2 &= TS^{-3}\alpha_1 = TS^{-3}TS^{-2}\alpha_0 = [\overline{2; 1, 1, 4}], \\ \alpha_3 &= TS^{-2}\alpha_2 = [\overline{1; 1, 4, 2}], \\ \alpha_4 &= TS^{-1}\alpha_3 = [\overline{1; 4, 2, 1}], \text{and} \\ \alpha_5 &= TS^{-1}\alpha_4 = [\overline{4; 2, 1, 1}]. \end{aligned}$$

6 SIMPLE FORMS AND NUMBERS

In this section we define simple λ-BQFs and simple numbers, which are easily characterized and are related to reduced forms and numbers. We use the function Φ_p, defined in Section 1, to put the simple forms in each hyperbolic equivalence class into a cycle.

We call a hyperbolic λ_p-BQF $Q = [A, B, C]$ *G_p-simple* if $A > 0 > C$. If Q is a simple form, we will say that α_Q is a G_p-simple number.

Lemma 10. *Let $p \geq 3$ and put $\lambda = \lambda_p$. Suppose that $Q = [A, B, C]$ is a hyperbolic λ-BQF associated with $\alpha = \alpha_Q$. Then Q is simple if and only if $\alpha' < 0 < \alpha$.*

Proof. The proof is an exercise in calculating with inequalities. $\square$

Next we establish the connection between reduced and simple numbers.

Theorem 3. *Let $p \geq 3$ and put $\lambda = \lambda_p$. Suppose that α is a G_p-simple number. Then $S_\lambda^n \alpha$, $n = -\left[\frac{\alpha'}{\lambda}\right]$ is a G_p-reduced number. The set of G_p-simple numbers is given by*

$$\mathcal{Z} = \left\{ S_\lambda^{-i}\beta \mid \beta \text{ is } G_p\text{–reduced, } 1 \leq i \leq \left[\frac{\beta}{\lambda}\right] \right\}.$$

Proof. Put $S = S_\lambda$ and let α be a G_p-simple number. Then $\alpha' < 0 < \alpha$, so $n = -\left[\frac{\alpha'}{\lambda}\right] \geq 1$ and $S^n\alpha > \lambda$. Also, $-n < \frac{\alpha'}{\lambda} < 1-n$ implies that $0 < S^n\alpha' < \lambda$. Thus $0 < S^n\alpha' < \lambda < S^n\alpha$, and $\beta = S^n\alpha$ is a reduced number by Theorem 1.

To prove the second statement, we first let α be any G_p-simple number, so $\alpha' < 0 < \alpha$. Then $\beta = S^i\alpha$, $i = -\left[\frac{\alpha'}{\lambda}\right]$, is reduced and $\left[\frac{\beta}{\lambda}\right] = \left[\frac{\alpha}{\lambda}\right] + i \geq i$. Now $\left[\frac{\alpha'}{\lambda}\right] \leq -1$, so $i \geq 1$, and $\alpha \in \mathcal{Z}$.

To finish proving the second statement, we suppose that $\alpha \in \mathcal{Z}$. Then $\alpha = S^{-i}\beta$, $1 \leq i \leq \left[\frac{\beta}{\lambda}\right]$, for some reduced β. Now $i \leq \left[\frac{\beta}{\lambda}\right]$ implies that $\alpha = S^{-i}\beta > 0$. The fact that $i \geq 1$, along with $\beta' < \lambda$, implies that $\alpha' = S^{-i}\beta' < 0$. Thus $\alpha' < 0 < \alpha$ and α is simple. □

We restate Theorem 3 in terms of hyperbolic λ-BQFs.

Corollary. *Let $p \geq 3$ and put $\lambda = \lambda_p$. Every simple λ-BQF Q is transformed into a reduced form by a single application of S_λ^{-n}, with $n = -\left[\frac{\alpha'_Q}{\lambda}\right]$. The set of simple λ-BQFs is given by*

$$\left\{ Q \circ S_\lambda^i \mid Q \text{ is } G_p\text{–reduced, } 1 \leq i \leq \left[\frac{\beta_Q}{\lambda}\right] \right\}.$$

Since there are a finite number of reduced λ-BQFs of a given discriminant, the Corollary implies that there are also a finite number of simple forms of a given discriminant. Also, every reduced form Q (or number α) is connected by S^j to $\left[\frac{\beta_Q}{\lambda}\right]$ simple forms (numbers). In particular, if $\beta_Q < \lambda$ then $Q \circ S^j$ fails to be simple for any j.

Simple numbers (and associated forms) may also be put into cycles, using the function Φ_p defined in Section 1.

Theorem 4. *Let $p \geq 3$ and put $\lambda = \lambda_p$. The finite orbits of Φ_p are the set $\{0\}$ and the sets*

$$\mathcal{Z}_\mathcal{A} = \{\alpha_Q \mid Q \in \mathcal{A}, \text{ simple}\},$$

where $\mathcal{A}$ runs over all hyperbolic G_p-equivalence classes of λ-BQFs.

Proof. Put $S = S_\lambda$ and $U = U_\lambda$. Any finite orbit except $\{0\}$ has the form

$$\mathcal{C} = \{\alpha_1, \alpha_1 - \lambda, \ldots, \alpha_1 - m_1\lambda, \alpha_2, \alpha_2 - \lambda, \ldots, \\ \alpha_2 - m_2\lambda, \ldots, \alpha_s, \alpha_s - \lambda, \ldots, \alpha_s - m_s\lambda\},$$

for some positive real numbers $\alpha_1, \alpha_2, \ldots, \alpha_s$, where $m_j = \left[\frac{\alpha_j}{\lambda}\right] \geq 0$, $1 \leq j \leq s$. We must also have that

$$\alpha_{j+1} = TU^{i_j}\left(\alpha_j - m_j\lambda\right),$$

$1 \leq j \leq s-1$, as well as

$$\alpha_1 = TU^{i_s}\left(\alpha_s - m_s\lambda\right),$$

with $1 \leq i_j \leq p-2$ for all j. Now since $m_j = \left[\frac{\alpha_j}{\lambda}\right]$, we have $0 < \alpha_j - m_j\lambda < \lambda$, so $U^{\ell_j+1}(0) < \alpha_j - m_j\lambda < U^{\ell_j}(0)$ for some ℓ_j, $2 \leq \ell_j \leq p-1$. We have eliminated the possibility that $\alpha_j - m_j\lambda = U^{\ell_j+1}(0)$, since then $\alpha_j - m_j\lambda$ would not be part of a finite orbit of Φ_p. Thus

$$\begin{aligned} \alpha_{j+1} &= \Phi_p(\alpha_j - m_j\lambda) \\ &= TU^{p-\ell_j}S^{-m_j}\alpha_j, \end{aligned}$$

$1 \leq j \leq s-1$, and

$$\begin{aligned} \alpha_1 &= \Phi_p(\alpha_s - m_s\lambda) \\ &= TU^{p-\ell_s}S^{-m_s}\alpha_s. \end{aligned}$$

For each j we put $\beta_j = S\alpha_j$ and $r_j = \left[\frac{\beta_j}{\lambda}\right] + 1 = m_j + 2 \geq 2$. Then

$$\begin{aligned} \beta_{j+1} &= S\alpha_{j+1} \\ &= U^{p-\ell_j+1}S^{-m_j}\alpha_j \\ &= U^{p-\ell_j+1}S^{-m_j-1}\beta_j, \end{aligned}$$

for $j \neq s$, and

$$\beta_0 = U^{p-\ell_s+1}S^{-m_s-1}\beta_s.$$

Reversing direction and using $m_j = r_j - 2$, we have

$$\beta_j = S^{r_j}TU^{\ell_j-2}\beta_{j+1},$$

$j \neq s$, and

$$\beta_s = S^{r_s}TU^{\ell_s-2}\beta_0.$$

Therefore

$$\beta_j = [\overline{r_0; \underbrace{1, \ldots, 1}_{\ell_j-2}, r_{j+1}, \ldots}],$$

$j \neq s$, and

$$\beta_s = [\overline{r_s; \underbrace{1, \ldots, 1}_{\ell_s-2}, r_0, \ldots}].$$

Now $0 \leq \ell_j - 2 \leq p-3$, so the λ-CF expansions are all admissible and β_j is G_p-reduced for each j. Clearly, the β_j are all part of the same cycle of reduced numbers in a G_p-equivalence class of numbers. We may let $\mathcal{A}$ represent the

equivalence class of λ-BQFs containing the corresponding forms. If any ones occur in the λ-CF expansions, the cycle contains reduced numbers other than the β_j. But these other reduced numbers are of the form $\beta = \overline{[1; \ldots]} < \lambda$, and are not connected with any G_p-simple numbers. Thus all of the simple numbers associated with forms in $\mathcal{A}$ are of the form $S^{-i}\beta_j$, $1 \leq j \leq s$, where i is a positive integer. By Theorem 3,

$$\begin{aligned}
\mathcal{Z}_{\mathcal{A}} &= \left\{ S^{-i}\beta_j \mid 1 \leq i \leq \left[\frac{\beta_j}{\lambda}\right] \right\}_{j=1}^{s} \\
&= \left\{ S^{-(i-1)}\alpha_j \mid 0 \leq i-1 \leq m_i \right\}_{j=1}^{s} \\
&= \mathcal{C}.
\end{aligned}$$

□

Example 2. In Example 1 we found the reduced λ_5-BQFs in a G_5-equivalence class $\mathcal{A}$. The forms

$$Q_{\alpha_2} = [3\lambda + 4, -11\lambda - 3, \lambda + 2],$$

and

$$Q_{\alpha_5} = [\lambda + 2, -13\lambda - 5, 11\lambda + 8],$$

are the only reduced forms $Q = [A, B, C]$ in $\mathcal{A}$ which correspond to reduced numbers greater than λ. The simple forms in $\mathcal{A}$ are all related to Q_{α_2} and Q_{α_5} by S^j as

$$\begin{aligned}
Q_{\alpha_2} \circ S &= [3\lambda + 4, 3\lambda + 3, -3\lambda - 2], \\
Q_{\alpha_5} \circ S &= [\lambda + 2, -7\lambda - 3, -3\lambda - 2], \\
Q_{\alpha_5} \circ S^2 &= [\lambda + 2, -\lambda - 1, -9\lambda - 6], \text{and} \\
Q_{\alpha_5} \circ S^3 &= [\lambda + 2, 5\lambda + 1, -7\lambda - 4].
\end{aligned}$$

We could calculate the corresponding simple numbers directly from these λ_5-BQFs. Instead, we will use the reduced numbers we found in Example 1 along with Theorem 3. The simple numbers in $\mathcal{A}$ are

$$\begin{aligned}
\mathcal{Z}_{\mathcal{A}} &= \left\{ S^{-1}\alpha_2, S^{-1}\alpha_5, S^{-2}\alpha_5, S^{-3}\alpha_5 \right\} \\
&= \left\{ [1; \overline{1, 1, 4, 2}], [3; \overline{2, 1, 1, 4}], [2; \overline{2, 1, 1, 4}], [1; \overline{2, 1, 1, 4}] \right\} \\
&= \left\{ \frac{-3\lambda - 3 + \sqrt{D}}{6\lambda + 8}, \frac{7\lambda + 3 + \sqrt{D}}{2\lambda + 4}, \frac{\lambda + 1 + \sqrt{D}}{2\lambda + 4}, \frac{-5\lambda - 1 + \sqrt{D}}{2\lambda + 4} \right\} \\
&= \left\{ \alpha_1, \alpha_2, \alpha_3, \alpha_4 \right\},
\end{aligned}$$

where $D = 135\lambda + 86 = \frac{307+135\sqrt{5}}{2}$. Finally, these simple numbers form a finite

orbit of Φ_5. We have

$$\begin{aligned}\Phi_5(\alpha_1) &= TU\alpha_1 = \alpha_2,\\ \Phi_5(\alpha_2) &= TU^4\alpha_2 = \alpha_2 - \lambda = \alpha_3,\\ \Phi_5(\alpha_3) &= TU^4\alpha_3 = \alpha_3 - \lambda = \alpha_4,\\ \Phi_5(\alpha_4) &= TU^3\alpha_4 = \alpha_1.\end{aligned}$$

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